# Dirac Operators on $\mathbb{R}^n$ with singular potentials

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We consider the $n-$dimensional Dirac operators with singular $\delta-$ type potentials supported on hypersurfaces in $\mathbb{R}^n$. In dimensions 2 and 3, such operators arise in problems of penetration of relativistic particles through obstacles created by potentials with supports on curves and surfaces.
In a rigorous setting, formal Dirac operators with singular potentials are realized as unbounded operators in Hilbert spaces with domains described by interaction conditions on sets carrying singular potentials.

In the last decade, the study of the spectral properties of such operators in dim 2 and 3 has attracted a lot of attention of many authors. See: Arrizabalaga, N.,Mas,A.and Vega,L. (2014); R.D.Benguria, S. Fournais, E. Stockmeyer, H. Van Den Bosch(2017); Brasche,J.F.,Exner, N. Arrizabalaga, A. Mas, and L. Vega (2014); Behrndt,J.,Exner,P.,Holzmann,M.,Lotoreichik,V (2017); Behrndt,J.,Exner,P.,Holzmann,M.,Lotoreichik,V. (2018); Behrndt,J, Holzmann, M., Ourmières-Bonafos,T., Pankrashkin,K. (2020), Cassano,B., Lotoreichich,V., Mas,A., and Tusek,M (2021); Holzmannm, M (2021). There is extencive literaure devoted the boundary value problems for the Dirac operators and their applications in te index theory and geometry.

We consider the formal Dirac operators on $\mathbb{R}^n$

$$D_{\mathbf{A},\Phi,m,\Gamma\delta_\Sigma} = \mathfrak{D}_{\mathbf{A},\Phi,m} + \Gamma\delta_\Sigma \tag{1}$$

where

$$\begin{aligned} \mathfrak{D}_{\mathbf{A},\Phi,m} &= \sum_{j=1}^{n} \alpha_j (D_{x_j} + \mathbf{A}) + m\alpha_{n+1} + \Phi I_N \\ &= \boldsymbol{\alpha} \cdot (\mathbf{D} + \mathbf{A}) + m\alpha_{n+1} + \Phi I_N, D_{x_j} = -i\partial_{x_j} \end{aligned} \tag{2}$$

is the Dirac operator on $\mathbb{R}^n$ with a regular magnetic potential $A = (A_1, ..., A_n)$, electrostatic scalar potential $\Phi$, and with variable mass $m$ of particles. In formula (2) $\alpha_j, j = 1, ..., n+1$ are the $N \times N$ Dirac matrices, that is the Hermitian matrices satisfying the relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk} I_N, j, k = 1, ..., n+1. \tag{3}$$

where $I_N$ is the $N \times N$ unit matrix, $N = N(n) = 2^{[(n+1)/2]}$, $\Gamma\delta_\Sigma$ is the singular potential, $\Gamma = (\Gamma_{i,j})_{i,j=1}^N$ is $N \times N$ matrix and $\delta_\Sigma$ is the delta-function with the support on a uniformly regular $C^2-$hypersurface $\Sigma \subset \mathbb{R}^n$ being the common boundary of open sets $\Omega_\pm \subset \mathbb{R}^n$.
(The definition of uniformly regular hypersurface will be given below. It should be noted that this class contains both bounded and unbounded hypersurfaces.)
We assume that $A_j, \Phi, m$ belong to $L^\infty(\mathbb{R}^n)$, and $\Gamma_{i,j} \in C_b^1(\Sigma)$ the space of differentiable on $\Sigma$ functions bounded together with their derivatives. Important examples of singular potentials are the linear combination of electrostatic and Lorentz singular potentials

$$\Gamma = \eta\alpha_{n+1} + \tau I_N \text{ on } \Sigma \tag{4}$$

where the functions $\eta, \tau \in C_b^1(\Sigma)$.

Let $H^1(\Omega_\pm, \mathbb{C}^N)$ be the Sobolev spaces of distributions on $\Omega^\pm$ with values in $\mathbb{C}^N$ and $H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N) = H^1(\Omega_+, \mathbb{C}^N) \oplus H^1(\Omega_-, \mathbb{C}^N)$. We associate with the formal Dirac operator $D_{\mathbf{A},\Phi,m,\Gamma\delta_\Sigma}$ the unbounded in $L^2(\mathbb{R}^n, C^N)$ operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ defined by the Dirac operator $\mathfrak{D}_{\mathbf{A},\Phi,m}$ with domain defined by the interaction (transmission) condition

$$\begin{aligned} dom\ \mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} &= H^1_{\mathfrak{B}_\Sigma}(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N) \\ &= \left\{ \mathbf{u} \in H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N) : \mathfrak{B}_\Sigma \mathbf{u} = a_+ \gamma_\Sigma^+ \mathbf{u} + a_- \gamma_\Sigma^- \mathbf{u} = 0 \text{ on } \Sigma \right\} \end{aligned} \tag{5}$$

where $\gamma_\Sigma^\pm : H^1(\Omega^\pm, \mathbb{C}^N) \to H^{1/2}(\Sigma, \mathbb{C}^N)$ are the trace operators, and

$$a_\pm = \frac{1}{2}\Gamma \mp i\boldsymbol{\alpha} \cdot \boldsymbol{\nu} \quad \text{on } \Sigma, \tag{6}$$

$\alpha \cdot \nu = \sum_{j=1}^n \alpha_j \nu_j$, $\nu = (\nu_1, \dots, \nu_n)$ is the field of unit normal vectors to $\Sigma$ directed into $\Omega^-$.

**The first part** of the talk we discuss the self-adjointness in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ of unbounded operators $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ associated with the singular Dirac operator.

**The second part** is devoted to the essential spectrum of operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ if the interaction hypersurface $\Sigma$ is bounded or unbounded.

We say that $\Sigma \subset \mathbb{R}^n$ is a *uniformly regular* hypersurface if : (i) there exists $r > 0$ such that for each point $x_0 \in \Sigma$ there exists a ball $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ and a diffeomorphism $\varphi_{x_0} : B_r(x_0) \to B_1(0)$ such that

$$\varphi_{x_0}(B_r(x_0) \cap \Sigma) = B_1(0) \cap \mathbb{R}^{n-1}_{y'};$$

(ii) let $\varphi^i_{x_0}, \psi^i_{x_0}, i = 1, ..., n$ be the coordinate functions of the mappings $\varphi_{x_0}, \varphi^{-1}_{x_0}$. Then

$$\sup_{x_0 \in \Sigma} \sup_{|\alpha| \leq 2, x \in B_r(x_0)} \left| \partial^\alpha \varphi^i_{x_0}(x) \right| < \infty, \sup_{x_0 \in \Sigma} \sup_{|\alpha| \leq 2, x \in B_r(x_0)} \left| \partial^\alpha \psi^i_{x_0}(x) \right| < \infty, i = 1, ...$$

Note that compact closed $C^2-$hypersurfaces are uniformly regular automatically.

# Realization of Dirac operator with singular potentials as an unbounded operator

Let

$$\mathfrak{D}_{\mathbf{A},\Phi,m,\Gamma\delta}\mathbf{u}(x) = (\mathfrak{D}_{\mathbf{A},\Phi,m} + \Gamma\delta_\Sigma)\,\mathbf{u}(x), x \in \mathbb{R}^n$$

be the formal Dirac operator defined by formulas (1),(2). We assume that $\Sigma$ is the $C^2-$hypersurface in $\mathbb{R}^n$, $A_j, \Phi, m \in L^\infty(\mathbb{R}^n)$, $\Gamma = (\Gamma_{i,j})_{i,j=1}^N, \Gamma_{i,j} \in C_b^1(\Sigma)$.

We define the product $\Gamma\delta_\Sigma\mathbf{u}$ where $\mathbf{u} \in H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ as a distribution in $\mathcal{D}'(\mathbb{R}^n, \mathbb{C}^N) = \mathcal{D}'(\mathbb{R}^n) \otimes \mathbb{C}^N$ acting on the test functions $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$ as

$$\begin{aligned} (\Gamma\delta_\Sigma\mathbf{u})(\boldsymbol{\varphi}) &= \frac{1}{2}\int_\Sigma \Gamma(s)\left(\gamma_\Sigma^+\mathbf{u}(s) + \gamma_\Sigma^-\mathbf{u}(s)\right)\cdot\boldsymbol{\varphi}(s)ds, \qquad (7) \\ \mathbf{u}\cdot\mathbf{v} &= \sum_{j=1}^N u_j\cdot\bar{v}_j \end{aligned}$$

Formula (7) yields that in the distribution sense

$$D_{\mathbf{A},\Phi,m,\Gamma\delta_\Sigma}\mathbf{u} = \mathfrak{D}_{\mathbf{A},\Phi,m}\mathbf{u} - \left[ i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\left(\gamma_\Sigma^+\mathbf{u} + \gamma_\Sigma^-\mathbf{u}\right) - \frac{1}{2}\Gamma\left(\gamma_\Sigma^+\mathbf{u} - \gamma_\Sigma^-\mathbf{u}\right)\right]\delta_\Sigma \tag{8}$$

where $\mathfrak{D}_{\mathbf{A},\Phi,m}\mathbf{u}$ is the regular distribution given by the function $\mathfrak{D}_{\mathbf{A},\Phi,m}\mathbf{u} \in L^2(\mathbb{R}^n, \mathbb{C}^N)$.
It yields that $\mathfrak{D}_{\mathbf{A},\Phi,m,\Gamma\delta_\Sigma}\mathbf{u} \in L^2(\mathbb{R}^n, \mathbb{C}^N)$ if and only if

$$-i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\left(\gamma_\Sigma^+\mathbf{u} - \gamma_\Sigma^-\mathbf{u}\right) + \frac{1}{2}\Gamma\left(\gamma_\Sigma^+\mathbf{u} + \gamma_\Sigma^-\mathbf{u}\right) = 0 \text{ on } \Sigma. \tag{9}$$

Condition (9) can be written of the form

$$\mathfrak{B}_\Sigma\mathbf{u} = a_+\gamma_\Sigma^+\mathbf{u} + a_-\gamma_\Sigma^-\mathbf{u} = \mathbf{0} \text{ on } \Sigma \tag{10}$$

where $a_\pm$ are $N \times N$ matrices:

$$a_\pm = \frac{1}{2}\Gamma \mp i\boldsymbol{\alpha}\cdot\boldsymbol{\nu} \text{ on } \Sigma, \boldsymbol{\alpha}\cdot\boldsymbol{\nu} = \sum_{j=1}^n \alpha_j \nu_j \tag{11}$$

where $\mathbf{v} = (\nu_1, ..., \nu_n)$ is the field of unit normal vectors directed to $\Omega_-$.

We associate with the formal Dirac operator $D_{\mathbf{A},\Phi,\Gamma\delta_\Sigma}$ the unbounded in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ defined by the Dirac operator $\mathfrak{D}_{\mathbf{A},\Phi,m}$ with the domain $dom\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} = H^1_{\mathfrak{B}_\Sigma}(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ and the bounded operator of the interaction (transmission) problem

$$\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}\mathbf{u} = \left\{ \begin{array}{c} \mathfrak{D}_{\mathbf{A},\Phi,m}\mathbf{u} \text{ on } \mathbb{R}^n \diagdown \Sigma, \\ \mathfrak{B}_\Sigma \mathbf{u} = a_+\gamma_\Sigma^+ \mathbf{u} + a_-\gamma_\Sigma^- \mathbf{u} \text{ on } \Sigma \end{array} \right. \qquad (12)$$

acting from $H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$.

# Parameter-dependent interaction problems

Let

$$\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)\mathbf{u} = \left(\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} - i\mu I_N\right)\mathbf{u} \tag{13}$$

$$= \left\{ \begin{array}{c} \mathfrak{D}_{\mathbf{A},\Phi}(i\mu)\mathbf{u} = (\mathfrak{D}_{\mathbf{A},\Phi,m} - i\mu I_N)\mathbf{u} \text{ on } \mathbb{R}^n \diagdown \Sigma, \\ \mathfrak{B}_\Sigma \mathbf{u} = a_+\gamma_\Sigma^+ \mathbf{u} + a_-\gamma_\Sigma^- \mathbf{u} \text{ on } \Sigma \end{array} \right., \mu \in \mathbb{R}$$

be the operator of parameter-dependent interaction problem acting from $H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$.
We assume as above that
$A_j, j = 1, ..., n, \Phi, m \in L^\infty(\mathbb{R}^n), \Gamma_{ij} \in C_b^1(\Sigma), i, j = 1, ..., N$, where $\Sigma \subset \mathbb{R}^n$ is a $C^2-$uniformly regular hypersurface being the common boundary of domains $\Omega_\pm$. We consider the invertibility of the operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ for large values of $|\mu|$ .

(i) Since

$$\mathfrak{D}_{\alpha}^{0}(i\mu)\mathfrak{D}_{\alpha}^{0}(-i\mu) = (\boldsymbol{\alpha}\cdot\mathbf{D} - i\mu I_N)(\boldsymbol{\alpha}\cdot\mathbf{D} + i\mu I_N) = \left(-\Delta_n + \mu^2\right) I_N$$

the operator $\mathfrak{D}_{\mathbf{A},\Phi,m}(i\mu)$ is elliptic with parameter.
(ii) The main part of the parameter-dependent operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_{\Sigma}}(i\mu)$ is

$$\mathbb{D}_{\boldsymbol{\alpha},\mathfrak{B}_{\Sigma}}^{0}(i\mu)\mathbf{u}(x) = \left(\mathbb{D}_{\boldsymbol{\alpha},\mathfrak{B}_{\Sigma}}^{0} - i\mu I_N\right)\mathbf{u}(x) \tag{14}$$
$$= \begin{cases} \mathfrak{D}_{\boldsymbol{\alpha}}^{0}(i\mu)\mathbf{u} = (\alpha\cdot\mathbf{D}_x - i\mu I_N)\,\mathbf{u} \text{ on } \mathbb{R}^n\diagdown\Sigma \\ \mathfrak{B}_{\Sigma}\mathbf{u} = a_{+}\gamma_{\Sigma}^{+}\mathbf{u} + a_{-}\gamma_{\Sigma}^{-}\mathbf{u} \text{ on } \Sigma \end{cases}, \mu\in\mathbb{R},$$

where $a_{\pm} = \frac{1}{2}\Gamma \mp i\alpha\cdot\boldsymbol{\nu}$.

Passing to the local coordinate system at the point $x_0 \in \Sigma$, and freezing the coefficients, we obtain the interaction operator

$$\mathbb{D}_{\boldsymbol{\alpha},\mathfrak{B}_{\mathbb{R}^{n-1}}(x_0)}(i\mu)\mathbf{w}(y)$$
$$= \begin{cases} \mathfrak{D}^0_{\tilde{\boldsymbol{\alpha}}}(i\mu)\mathbf{w} = (\tilde{\alpha}\cdot\mathbf{D}_y - i\mu I_N)\,\mathbf{w} \text{ on } (\mathbb{R}^n_+ \cup \mathbb{R}^n_-) \\ \mathfrak{B}_{\mathbb{R}^{n-1}}(x_0)\mathbf{w} = \tilde{a}_+(x_0)\gamma^+_{\mathbb{R}^{n-1}_{y'}}\mathbf{w} + \tilde{a}_-(x_0)\gamma^-_{\mathbb{R}^{n-1}_{y'}}\mathbf{w} \text{ on } \mathbb{R}^{n-1}_{y'} \end{cases} \quad (15)$$

where $\mathbb{R}^{n-1}_{y'} = \{y \in \mathbb{R}^n_y : y_n = 0\}$, $\tilde{\alpha} = (\tilde{\alpha}_1, ..., \tilde{\alpha}_n)$ are new Dirac matrices, and $\tilde{a}_\pm(x_0) = \frac{1}{2}\Gamma(x_0) \mp i\tilde{\alpha}_n$, and $\mathbf{w} \in H^1(\mathbb{R}^n_+, \mathbb{C}^n) \oplus H^1(\mathbb{R}^n_-, \mathbb{C}^n)$ with the discontinuity on $\mathbb{R}^{n-1}$.

We study the invertibility of the operator $\mathbb{D}_{\boldsymbol{\alpha},\mathfrak{B}_{\mathbb{R}^{n-1}}(x_0)}(i\mu)$ for $\mu \in \mathbb{R}\diagdown\{0\}$ . Applying the Fourier transform with respect to $y' \in \mathbb{R}^{n-1}$ we obtain the family of one-dimensional interaction problems

$$\mathbb{D}_{\tilde{\boldsymbol{\alpha}},\mathfrak{B}_{\mathbb{R}^{n-1}}(x_0)}(\xi', i\mu)\hat{\mathbf{w}}(z) = \left\{ \begin{array}{c} (\tilde{\boldsymbol{\alpha}}' \cdot \xi' - i\tilde{\alpha}_n \frac{d}{dz} - i\mu I_N)\hat{\mathbf{w}}(z), z \in \mathbb{R}\diagdown 0 \\ \mathfrak{B}_{\mathbb{R}^{n-1}}(x_0)\hat{\mathbf{w}}(z) \\ = \tilde{a}_+(x_0)\gamma_0^+\hat{\mathbf{w}}(0) + \tilde{a}_-(x_0)\gamma_0^-\hat{\mathbf{w}}(0) \end{array} \right. , \tag{16}$$

acting from $H^1(\mathbb{R}\diagdown 0, \mathbb{C}^N)$ to $L^2(\mathbb{R}, \mathbb{C}^N) \oplus \mathbb{C}^N$ where $\xi' \in \mathbb{R}^{n-1}$.
Note that $\mathbb{D}_{\boldsymbol{\alpha},\mathfrak{B}_{\mathbb{R}^{n-1}}(x_0)}(i\mu, \xi')$ is invertible for $(\xi', \mu) \in \mathbb{S}^{n-1}$ if and only if

$$\ker \mathbb{D}_{\boldsymbol{\alpha},\mathfrak{B}_{\mathbb{R}^{n-1}}(x_0)}(i\mu, \xi') = \{0\} .$$

The homogeneous equations on the half-axis $\mathbb{R}_\pm = \{z \in \mathbb{R} : z \gtrless 0\}$

$$(\tilde{\boldsymbol{\alpha}}' \cdot \xi' - i\tilde{\alpha}_n \frac{d}{dz} - i\mu I_N)\hat{\mathbf{w}}_\pm(\mu, \xi', z) = \mathbf{0}, z \gtrless 0. \quad (17)$$

has exponentially decreasing solutions

$$w_\pm(\xi', z, \mu) = \frac{\Theta_\pm(\xi', \mu)}{2\rho}\hat{\mathbf{c}}_\pm(\xi', \mu)e^{\mp\rho z}, z \gtrless 0, \rho = \sqrt{|\xi'|^2 + \mu^2} \quad (18)$$

where

$$\begin{aligned} \Theta_\pm(\xi', \mu) &= i\mu I_N + \tilde{\alpha}' \cdot \xi' \pm i\rho\tilde{\alpha}_n, \\ \hat{\mathbf{c}}_\pm(\xi', \mu) &= \sum_{j=1}^{N} \hat{c}_\pm^j(\xi', \mu)\mathbf{e}_j, \hat{c}_\pm^j(\xi', \mu) \in \hat{H}^{1/2}(\mathbb{R}^{n-1}) \end{aligned}$$

and $\{\mathbf{e}_j\}_{j=1}^N$ is a base of $\mathbb{C}^N$. Note that *rank* $\Theta_\pm(\xi', \mu) = \frac{N}{2}$ for every $(\xi', \mu) \in \mathbb{S}^{n-1}$.

There exists a sub-systems $\left\{\mathbf{e}_j^{\pm}\right\}_{j=1}^{N/2}$ of $\{\mathbf{e}_j\}_{j=1}^{N}$ such that the vectors

$$\left\{\mathbf{h}_j^{\pm}(\xi',\mu)=\frac{\Theta_{\pm}(\xi',\mu)}{2\rho}\mathbf{e}_j^{\pm}\right\}_{j=1}^{N/2}$$

are linear independent. Hence, the general exponentially decreasing solutions of the homogeneous equations (17) are

$$\tilde{w}_{\pm}(\xi',z,\mu)=\left(\sum_{j=1}^{N/2}\hat{c}_{\pm}^{j}(\xi',\mu)\mathbf{h}_j^{\pm}\left(\xi',\mu\right)\right)e^{\mp\rho z}, z\gtrless 0 \qquad (19)$$

where $\hat{c}_{\pm}^{j}(\xi',\mu)\in\hat{H}^{1/2}(\mathbb{R}_{\xi'}^{n-1})$ are arbitrary functions.

Substituting $\tilde{w}_{\pm}(\xi', z, \mu)$ into the interaction conditions we obtain the system of $N$ linear equations

$$\sum_{j=1}^{N/2} \left( \hat{c}_{+}^{j}(\xi', \mu)\tilde{a}_{+}(x_0)\mathbf{h}_j^{+}(\xi', \mu) + \hat{c}_{-}^{j}(\xi', \mu)\tilde{a}_{-}(x_0)\mathbf{h}_j^{-}(\xi', \mu) \right) = 0 \quad (20)$$

with respect to

$$(\hat{c}_{+}^{1}(\xi', \mu), ..., \hat{c}_{+}^{N/2}(\xi', \mu), \hat{c}_{-}^{1}(\xi', \mu), ..., \hat{c}_{-}^{N/2}(\xi', \mu)) \in \mathbb{C}^N.$$

with the $N \times N-$matrix $\mathfrak{M}_{x_0}(\xi', \mu)$

$$\begin{aligned} \mathfrak{M}_{x_0}(\xi', \mu) \quad = \quad & (\tilde{a}_{+}(x_0)\mathbf{h}_1^{+}(\xi', \mu), ..., \tilde{a}_{+}(x_0)\mathbf{h}_{N/2}^{+}(\xi', \mu), \\ & \tilde{a}_{-}(x_0)\mathbf{h}_1^{-}(\xi', \mu), ..., \tilde{a}_{-}(x_0)\mathbf{h}_{N/2}^{-}(\xi', \mu)). \end{aligned} \quad (21)$$

We say that the operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ satisfies: (i) the local parameter-dependent Lopatinsky-Shapiro condition at the point $x_0 \in \Sigma$ if

$$\det \mathfrak{M}_{x_0}(\xi', \mu) \neq 0 \text{ for every } \ (\xi', \mu) \in \mathbb{S}^{n-1}; \tag{22}$$

ii) the uniform parameter-dependent Lopatinsky-Shapiro conditions if

$$\inf_{x\in\Sigma,\ (\xi',\mu)\in\mathbb{S}^{n-1}} \left|\det \mathfrak{M}_x(\xi', \mu)\right| > 0. \tag{23}$$

## Theorem

*Let: (i) $A_j, j = 1, ..., n$, $\Phi, m \in L^\infty(\mathbb{R}^n)$, (ii) $\Sigma \subset \mathbb{R}^n$ be the uniformly regular $C^2-$hypersurface, the matrix $\Gamma = (\Gamma_{kl})_{k,l=1}^N \in C_b^1(\Sigma, \mathcal{B}(\mathbb{C}^N))$, (iii) the uniform parameter-dependent Lopatinsky-Shapiro conditions (23) be satisfied. Then there exits $\mu_0 > 0$ such that the operator*

$$\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu) : H^1(\mathbb{R}^n \diagup \Sigma, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$$

*is invertible for all $\mu \in \mathbb{R} : |\mu| \geq \mu_0$.*

The proof of the theorem uses a countable partition of the unity of finite multiplicity, which exists for an unbounded hypersurface $\Sigma$ due to its uniform regularity.

## Corollary

*Let the conditions of Theorem 1 be satisfied. Then there exists a constant* $C > 0$ *such that for every function* $\mathbf{u} \in H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ *the a priori estimate*

$$\|\mathbf{u}\|_{H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)} \leq C \left( \|\mathfrak{D}_{\mathbf{A},\Phi,m}\mathbf{u}\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \|\gamma_\Sigma \mathbf{u}\|_{H^{1/2}(\Sigma,\mathbb{C}^N)} + \|\mathbf{u}\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \right) \tag{24}$$

*holds.*

Let $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ be an unbounded operator $L^2(\mathbb{R}^n,\mathbb{C}^N)$ associated with the formal Dirac operators (1),(2) which is defined by the Dirac operator

$$\mathfrak{D}_{\mathbf{A},\Phi,m} = \boldsymbol{\alpha}\cdot(\mathbf{D}_x+\mathbf{A})+\alpha_{n+1}m+\Phi I_N$$

with the domain

$$\begin{aligned} & dom\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} = H^1_{\mathfrak{B}_\Sigma}(\mathbb{R}^n\diagdown\Sigma,\mathbb{C}^N) \\ = & \left\{\mathbf{u}\in H^1(\mathbb{R}^n\diagdown\Sigma,\mathbb{C}^N) : \mathfrak{B}_\Sigma\mathbf{u} = a_+\gamma_\Sigma^+\mathbf{u}+a_-\gamma_\Sigma^-\mathbf{u} = \mathbf{0} \text{ on } \Sigma\right\}, \\ a_\pm = & \frac{1}{2}\Gamma \mp i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}. \end{aligned}$$

## Theorem

*Let* $\Sigma \subset \mathbb{R}^n$ *be the uniformly regular hypersurface, the vector potential* $\mathbf{A} \in L^\infty(\mathbb{R}^n, \mathbb{R}^N)$, *scalar potentials* $\Phi$, *and the variable mass* $m \in L^\infty(\mathbb{R}^n)$ *be real-valued, and* $\Gamma = (\Gamma_{ij})_{i,j=1}^N$ *be an Hermitian matrix with elements* $\Gamma_{ij} \in C_b^1(\Sigma)$. *Moreover, we assume that the uniform Lopatinsky-Shapiro condition (23) is satisfied. Then the operator* $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ *is self-adjoint in* $L^2(\mathbb{R}^n, \mathbb{C}^N)$.

## Proof.

The uniform Lopatinsky-Shapiro condition (23) yields a priori estimate (24) which implies that the operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is closed. Let $\mathbf{u},\mathbf{v}\in dom\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$. Integrating by parts we obtain that

$$\begin{aligned}&\left\langle \mathfrak{D}_{\mathbf{A},\Phi,m}\mathbf{u},\mathbf{v}\right\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}-\left\langle \mathbf{u},\mathfrak{D}_{\mathbf{A},\Phi,m}\mathbf{v}\right\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\\&=-\frac{1}{4}\left\langle\Gamma\left(\gamma_\Sigma^+\mathbf{u}-\gamma_\Sigma^-\mathbf{u}\right),\gamma_\Sigma^+\mathbf{v}-\gamma_\Sigma^-\mathbf{v}\right\rangle_{L^2(\Sigma,\mathbb{C}^N)}\\&+\frac{1}{4}\left\langle\gamma_\Sigma^+\mathbf{u}-\gamma_\Sigma^-\mathbf{u},\Gamma(\gamma_\Sigma^+\mathbf{v}-\gamma_\Sigma^-\mathbf{v}\right\rangle_{L^2(\Sigma,\mathbb{C}^N)}.\end{aligned}\tag{25}$$

Since the matrix $\Gamma$ is Hermitian the right side part in (25) is 0. Hence $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is symmetric.
We consider the operator $\mathcal{D}-i\mu I$ where $\mu\in\mathbb{R}$. If $|\mu|$ is large enough the operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}-i\mu I : H^1(\mathbb{R}^n\diagdown\Sigma,\mathbb{C}^n)\rightarrow L^2(\mathbb{R}^n,\mathbb{C}^N)$ is invertible. It implies that the defficiency numbers of $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ are 0. Hence the operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is self-adjoint. □

Let $\Gamma = \eta I_N + \tau \alpha_{n+1}$ where $\eta, \tau \in C_b^1(\Sigma)$ are real-valued functions. Let

$$\inf_{s \in \Sigma} \left| \eta^2(s) - \tau^2(s) - 4 \right| > 0. \tag{26}$$

Then the uniform Lopatinsky-Shapiro condition is satisfied . Ley the potentials $\mathbf{A}, \Phi, m$ be real-valued and condition (26) holds. Then by Theorem 3 the operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is self-adjoint if the potentials $\mathbf{A}, \Phi, m$ are real-valued and condition (26) holds.

Condition (26) of self-adjointnes has been obtained for dim 2, 3 by different methods in the papers: Behrndt,J.,Exner,P.,Holzmann,M.,Lotoreichik,V: On Dirac operators in $\mathbb{R}^3$ with electrostatic and Lorentz scalar $\delta$-shell interactions, Quantum Stud.: Math. Found., https://doi.org/10.1007/s40509-019-00186-6, (2019)

Behrndt,J, Holzmann, M., Ourmières-Bonafos,T., Pankrashkin,K.: Two-dimensional Dirac operators with singular interactions supported on closed curves Journal of Functional Analysis, V. 279, Is. 8, 108700 (2020)

# Splitting of the interaction conditions

For some types of obstacles, the penteration of particles is physically forbidden, and mathematically, penetration problems are divided into two boundary value problems. Such boundary value problems, describing the confinement of hadrons in a bounded volume, were first considered in the physical literature (A. Chodos, RL Jaffe, K. Johnson, CB Thorn, VF Weisskopf (1974), A. Chodos (1975), Johnson K (1975) These problems are called the MIT bag models (MIT - the Massachusetts Institute of Technology, where this hadronic model was first proposed).

We consider the interaction problem

$$\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}\mathbf{u} = \begin{cases} \mathfrak{D}_{\mathbf{A},\Phi,m}\mathbf{u} = \mathbf{f}, \text{ on } \mathbb{R}^n \diagdown \Sigma, \\ \mathfrak{B}_\Sigma \mathbf{u} = a_+ \gamma_\Sigma^+ \mathbf{u} + a_- \gamma_\Sigma^- \mathbf{u} = \boldsymbol{\varphi} \text{ on } \Sigma \end{cases} \tag{27}$$

where $a_\pm(s) = \frac{1}{2}\Gamma(s) \mp i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}(\mathbf{s})$. The interaction condition $\mathfrak{B}_\Sigma \mathbf{u} = \boldsymbol{\varphi}$ can be written as

$$\mathcal{P}^+ \gamma_\Sigma^+ \mathbf{u} + \mathcal{P}^- \gamma_\Sigma^- \mathbf{u} = i\left(\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\right)\boldsymbol{\varphi} \text{ on } \Sigma \tag{28}$$

where

$$\mathcal{P}^\pm = \frac{1}{2}(I_N \pm M), \text{ and } M = \frac{i}{2}\left(\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\right)\Gamma \text{ on } \Sigma. \tag{29}$$

Assume that

$$M^2 = I_N. \tag{30}$$

Then

$$\left(\mathcal{P}^{\pm}\right)^2 = \frac{1}{4}(I_N \pm M_\Gamma)^2 = \frac{1}{4}\left(I_N \pm 2M + M^2\right) = \mathcal{P}^{\pm},$$
$$\mathcal{P}^{+}\mathcal{P}^{-} = \frac{1}{4}(I_N - M^2) = 0.$$

Hence the operators $\mathcal{P}^{\pm}$ under condition (30) are the orthogonal projectors in $\mathbb{C}^N$ and interaction condition (27) splits into two independent boundary conditions

$$\mathcal{P}^{\pm}\gamma^{\pm}\mathbf{u} = \mathcal{P}^{\pm} i\left(\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\right)\boldsymbol{\varphi} \text{ on } \Sigma.$$

Hence in this case the interaction problem (27) splits into two boundary problems for Dirac operator

$$\mathbb{D}_{\mathbf{a},\Phi,m,\mathcal{P}^{\pm}}\mathbf{u}_{\pm} = \left\{ \begin{array}{c} \mathfrak{D}_{\mathbf{a},\Phi,m}\mathbf{u}_{\pm} = \mathbf{f}_{\pm} \text{ on } \Omega_{\pm}, \\ \mathcal{P}^{\pm}\gamma_{\Sigma}^{\pm}\mathbf{u}_{\pm} = \mathcal{P}^{\pm} i\left(\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\right)\boldsymbol{\varphi} \text{ on } \Sigma \end{array} \right. . \tag{31}$$

## Example

Let $\Gamma = \eta I_N + \tau \alpha_{n+1}, \eta, \tau \in \mathbb{R}$. Then

$$M = \frac{i}{2} (\boldsymbol{\alpha} \cdot \boldsymbol{\nu}) \Gamma = \frac{i}{2} (\boldsymbol{\alpha} \cdot \boldsymbol{\nu}) (\eta I_N + \tau \alpha_{n+1})$$

and $M^2 = -\frac{1}{4}(\eta^2 - \tau^2) I_N$. Hence $M^2 = I_N$ if $\eta^2 - \tau^2 = -4$. Under this condition the interaction problem (27) splits into the orthogonal sum of the boundary problems

$$D^{\pm}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} \mathbf{u} = \left\{ \begin{array}{c} \mathfrak{D}_{\mathbf{A},\Phi,m} \mathbf{u} \text{ on } \Omega_{\pm}, \\ \mathcal{P}^{\pm} \gamma^{\pm}_{\Sigma} \mathbf{u} = \mathcal{P}^{\pm} i (\boldsymbol{\alpha} \cdot \boldsymbol{\nu}) \boldsymbol{\varphi} \text{ on } \Sigma \end{array} \right. . \tag{32}$$

If $\eta = 0$ and $\tau^2 = 4$ the boundary value problems (32) for $n = 3, N = 4$ are called the *MIT bag problems* and they describe the confinement of particles in domains bounded by the hypersurfaces $\Sigma$.

# Fredholmness and essential spectrum of interaction operators

We consider the Fredholm theory of the interaction operators

$$\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}\mathbf{u} = \begin{cases} \mathcal{D}_{\mathbf{A},\Phi,m}\mathbf{u} \text{ on } \mathbb{R}^n \diagdown \Sigma \\ \mathfrak{B}_\Sigma \mathbf{u} = a_+\gamma_\Sigma^+\mathbf{u} + a_-\gamma_\Sigma^-\mathbf{u} \text{ on } \Sigma \end{cases} \quad (33)$$

acting from $H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$ and the essential spectrum of unbounded operators $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$. We consider here two types of the interaction problems:
$(a)$ $\Sigma$ is a closed $C^2-$hypersurface; $(b)$ $\Sigma$ is a $C^2-$hypersurface which are slowly oscillating at infinity, $(c)$ $A_j, \Phi, m \in SO^1(\mathbb{R}^n)$ where

$$SO^1(\mathbb{R}^n) = \left\{ a \in C_b^1(\mathbb{R}^n) : \lim_{x\to\infty} \partial_{x_j} a(x) = 0 \right\}$$

is the class of slowly oscillating at infinity functions. $(d)$ The Lopatinsky-Shapiro condition

$$\det \mathfrak{M}_x(\xi', 0) \neq 0 \text{ if } \left|\xi'\right| = 1 \quad (34)$$

holds at each point $x \in \Sigma$.

# Local principle

Cut-off functions: Let $\psi \in C_0^\infty(B_1(0))$, and $\psi(x) = 1$ for $x \in B_{1/2}(0)$, $0 \leq \psi(x) \leq 1$, $\chi(x) = 1 - \psi(x)$, $\psi_R(x) = \psi(x/R)$, $\chi_R(x) = \chi(x/R)$, $R > 0$.

## Definition

Let $\mathbf{X} = H^1(\mathbb{R}^n \diagup \Sigma, \mathbb{C}^N)$, $\mathbf{Y} = L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$. We say that the operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} : \mathbf{X} \to \mathbf{Y}$ is locally *Fredholm* at the point $x_0 \in \mathbb{R}^n$ if there exist a ball $B_\varepsilon(x_0)$, $\varepsilon > 0$ and operators $\mathcal{L}_{x_0}, \mathcal{R}_{x_0} \in \mathcal{B}(\mathbf{Y}, \mathbf{X})$ such that for every function $\varphi \in C_0^\infty(B_\varepsilon(x_0))$

$$\mathcal{L}_{x_0}\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}\varphi I_{\mathbf{X}} = \varphi I_{\mathbf{X}} + K'_{x_0},$$
$$\varphi\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}\mathcal{R}_{x_0} = \varphi I_{\mathbf{Y}} + K''_{x_0},$$

where $K'_{x_0} \in \mathcal{K}(\mathbf{X})$, $K''_{x_0} \in \mathcal{K}(\mathbf{Y})$;

## Definition

We say that the operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} : \mathbf{X} \to \mathbf{Y}$ is locally invertible at infinity if there exists $R > 0$ and operators $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}(\mathbf{Y}, \mathbf{X})$ such that

$$\begin{aligned} \mathcal{L}_R \mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} \chi_R I_{\mathbf{X}} &= \chi_R I_{\mathbf{X}}, \\ \chi_R \mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} \mathcal{R}_R &= \chi_R I_{\mathbf{Y}}. \end{aligned} \tag{35}$$

## Lemma

*(Local Principle) The operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} : \mathbf{X} \to \mathbf{Y}$ is a Fredholm operator if and only if $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is a locally Fredholm operator at every point $x \in \mathbb{R}^n$ and $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is locally invertible at infinity.*

Let $\Sigma \subset \mathbb{R}^n$ be a closed $C^2-$hypersurface. Since the Dirac operator is elliptic, and local Lopatinsky-Shapiro condition (34) is satisfied at every point $x \in \Sigma$, the operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is a Fredholm operator if and only if $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is locally invertible at infinity. The operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ coincides with $\mathfrak{D}_{\mathbf{A},\Phi,m}$ outside a some ball $B_0(R)$. It implies that $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is locally invertible at infinity if and only if the Dirac operator $\mathfrak{D}_{\mathbf{A},\Phi,m}$ has this property.

Let the function $a \in C_b^1(\mathbb{R}^n)$ and $\mathbb{Z}^n \ni h_k \to \infty$. We consider the sequence $\{a(\cdot + h_k)\}_{k \in \mathbb{N}}$. Applying the Arzelá-Ascoli Theorem one can find a subsequence $\{a(\cdot + h_{k_l})\}_{l \in \mathbb{N}}$ convergent to a limit function $a^h \in C_b(\mathbb{R}^n)$ uniformly on every compact set $K \subset \mathbb{R}^n$. If $a \in SO^1(\mathbb{R}^n)$ then the limit function $a^h$ is a constant.
Hence if $A_j, \Phi, m \in SO^1(\mathbb{R}^n)$ then every sequence $\mathbb{Z}^n \ni h_k \to \infty$ has a subsequence $h_{k_l}$ such that there exist limits

$$\mathbf{A}(x + h_{k_l}) \to \mathbf{A}^h, \Phi(x + h_{k_l}) \to \Phi^h, m(x + h_{k_l}) \to m^h$$

in the sense of uniformly convergence on the compact sets in $\mathbb{R}^n$ where $\mathbf{A}^h \in \mathbb{C}^n, \Phi^h, m^h \in \mathbb{C}$.
The operator $\mathfrak{D}_{\mathbf{A}^h, \Phi^h, m^h}$ is called a limit operator defined by the sequence $h_k \to \infty$.

## Theorem

*Let $\Sigma \subset \mathbb{R}^n$ be a closed $C^2$-hypersurface, $A_j, \Phi, m \in SO^1(\mathbb{R}^n)$, and the local Shapiro-Lopatinsky condition be satisfied at every point $x \in \Sigma$. Then the interaction operator $\mathbb{D}_{\mathbf{A},\mathbf{\Phi},\mathbf{m},\mathfrak{B}} : H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$ is Fredholm if and only if all limit operators $\mathfrak{D}_{\mathbf{A}^h,\Phi^h,m^h}$ are invertible.*

Since $\mathfrak{D}_{\mathbf{A}^h,\Phi^h,m^h}$ are operators with constant coefficients the condition of invertibility of $\mathfrak{D}_{\mathbf{A}^h,\Phi^h,m^h}$ is

$$\boldsymbol{\alpha} \cdot \xi + m^h \alpha_{n+1} + \Phi^h \neq 0 \text{ for all } \xi \in \mathbb{R}^n,$$

$\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n), \alpha_{n+1}$ are the Dirac matrices, $\boldsymbol{\alpha} \cdot \xi = \alpha_1 \xi_1 + ... + \alpha_n \xi_n$.

## Corollary

*Let $\Sigma \subset \mathbb{R}^n$ be a closed $C^2$-hypersurface, $A_j, \Phi, m \in SO^1(\mathbb{R}^n)$ be real-valued functions and the local Shapiro-Lopatinsky condition be satisfied at every point $x \in \Sigma$. Then*

$$\begin{aligned} & sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}} \\ = & \left(-\infty, \limsup_{x\to\infty}(\Phi(x) - |m(x)|)\right] \bigcup \left[\liminf_{x\to\infty}(\Phi(x) + |m(x)|), +\infty\right). \end{aligned} \tag{36}$$

*Formula (**??**) yields that*

$$sp_{dis}\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}} \subset \left(\limsup_{x\to\infty}(\Phi(x) - |m(x)|), \liminf_{x\to\infty}(\Phi(x) + |m(x)|\right).$$

**Proof.**

The above theorem implies that

$$sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}}=\bigcup_{h}sp\mathfrak{D}_{\mathbf{A}^{h},\Phi^{h},m^{h}} \tag{37}$$

Note that

$$sp\mathfrak{D}_{\mathbf{A}^{h},\Phi^{h},m^{h}}=\left(-\infty,\Phi^{h}-\left|m^{h}\right|\right]\bigcup\left[\Phi^{h}+\left|m^{h}\right|,+\infty\right). \tag{38}$$

Then (37) and (38) imply (36). □

We say that the unbounded $C^2-$hypersurface $\Sigma$ is slowly oscillating at infinity if for $R>0$ large enough

$$\Sigma_R = \Sigma \cap \mathbb{B}_R^n = \left\{ \begin{array}{c} x = (x', x_n) \in \mathbb{R}^n : x_n = f(x'), \\ x' = (x_1, ..., x_{n-1}) \in \mathbb{B}_R^{n-1} \end{array} \right\} \quad (39)$$

where $\mathbb{B}_R^n = \{x \in \mathbb{R}^n : |x| > R\}$, $f \in C^2(\mathbb{B}_R^{n-1})$ is a real-valued function, such that

$$\partial_{x_j} f \in C_b(\mathbb{R}^{n-1}), \lim_{x' \to \infty} \partial^2_{x_j x_k} f(x') = 0, j, k = 1, .., n-1. \quad (40)$$

Example.

$$f(x') = a\left|x'\right|^{\beta} \sin \log^{\alpha}\left|x'\right|, \left|x'\right| \geq R, a \in \mathbb{R}, 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1. \quad (41)$$

If $\beta = 1, \alpha = 0$ then $\Sigma$ is a conical at infinity hypersurface, and if $0 < \beta < 1$, then $\Sigma$ is a paraboloid, and if $0 < \alpha < 1$ then $\Sigma$ is an oscillating pertrubation of the cone or paraboloid.

We consider the Fredholm property of the operator

$$\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}\mathbf{u} = \begin{cases} \mathfrak{D}_{\mathbf{A},\Phi,m}\mathbf{u} \text{ on } \mathbb{R}^n \diagdown \Sigma \\ \mathfrak{B}_\Sigma \mathbf{u} = a_+\gamma_\Sigma^+\mathbf{u} + a_-\gamma_\Sigma^-\mathbf{u} \text{ on } \Sigma \end{cases} \tag{42}$$

under the assumptions:
$(i)$ the hypersurface $\Sigma$ is slowly oscillating at infinity;
$(ii)$ $a_j, \Phi, m \in SO(\mathbb{R}^n)$;
$(iii)$ $\Gamma = (\Gamma_{kl})_{k.l=1}^N$ with $\Gamma_{kl}(x', f(x')) \in SO(\mathbb{R}^{n-1})$;
$(iv)$ the local Lopatinsky-Shapiro condition is satisfied at every point $x \in \Sigma$.
Conditions $(i) - (iv)$ yield that $\mathbb{D}_{\mathbf{a},\Phi,m,\mathfrak{B}_\Sigma}$ is locally Fredholm operator at every point $x \in \mathbb{R}^n$. It implies that $\mathbb{D}_{\mathbf{a},\Phi,m,\mathfrak{B}_\Sigma}$ is a Fredholm operator if and only if this operator is locally invertible at infinity.

For simplicity we assume that

$$\Sigma = \left\{ x = (x', x_n) \in \mathbb{R}^n : x_n = f(x'), x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1} \right\}$$

and $\Sigma$ is the common boundary of the domains

$$\Omega_{\pm} = \left\{ x = (x', x_n) \in \mathbb{R}^n : x_n \gtrless f(x'), x' \in \mathbb{R}^{n-1} \right\}.$$

We denote by $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ the isomorphism given as

$$y = (y', y_n) = \Psi(x) = (x', x_n - f(x')), x = (x', x_n) \in \mathbb{R}^n.$$

Hence

$$\Psi(\Omega_{\pm}) = \mathbb{R}^n_{\pm} = \left\{ y = (y', y_n) \in \mathbb{R}^n : y_n \gtrless 0 \right\},$$
$$\Psi(\Sigma) = \left\{ y = (y', y_n) \in \mathbb{R}^n : y_n = 0 \right\} = \mathbb{R}^{n-1}.$$

Let

$$l_\omega = \left\{ x \in \mathbb{R}^n : x = t\omega, \omega \in \mathbb{S}^{n-1}, t > 0 \right\}$$

be a ray in $\mathbb{R}^n$, and $\Psi^{-1}(l_\omega)$ be a curve in $\mathbb{R}^n$. We denote by $\widetilde{\mathbb{R}^{n,\Psi}}$ the compactification of $\mathbb{R}^n$ which obtained by adjoining to each ray $\Psi^{-1}(l_\omega)$ the infinitely distant point $\eta_\omega$. If $\Psi = I$ we obtain the radial compactification $\widetilde{\mathbb{R}^n}$ isomorphic to the closed unit ball $\mathbb{B}_1$. We denote by $\widetilde{\Sigma^\Psi}$ the compactification of $\Sigma$ in the topology $\widetilde{\mathbb{R}^{n,\Psi}}$, and let $\mathbb{R}^n_{\Psi,\infty} = \widetilde{\mathbb{R}^{n,\Psi}} \diagdown \mathbb{R}^n$, and $\Sigma_{\Psi,\infty} = \widetilde{\Sigma^\Psi} \diagdown \Sigma$ be the corresponding sets of the infinitely distant points.

We introduce the families of limit operators defined by sequences tending to infinitely distant points.

a) Let $\mathbb{Z}^n \ni h_k \to \eta_\omega \in \mathbb{R}^n_{\Psi,\infty} \diagdown \Sigma_{\Psi,\infty}$ in the topology $\mathbb{R}^n$. Then the limit operators $\mathbb{D}^h_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ are of the form

$$\mathbb{D}^h_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} = \mathfrak{D}_{\mathbf{A}^h,\Phi^h,a^h} = \alpha \cdot (D + \mathbf{A}^h) + m^h \alpha_{n+1} + \Phi^h I_{\mathbb{N}}$$

where $\mathbf{A}^h = \lim_{k\to\infty} \mathbf{A}(h_k), \Phi^h = \lim_{k\to\infty} \Phi(h_k), m^h = \lim_{k\to\infty} m(h_k)$ are constant. Note that

$$sp\mathfrak{D}_{\mathbf{A}^h,\Phi^h,a^h} = (-\infty, -\left|m^h\right| + \Phi^h) \bigcup [\left|m^h\right| + \Phi^h, +\infty).$$

b) Let $\mathbb{Z}^n \ni h_k \to \eta_\omega \in \Sigma_{\Psi,\infty}$ in the topology $\mathbb{R}^n$. Then the limit operators defined by such sequences are operators of interaction on hyperplanes

$$\mathbb{D}^h_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}\mathbf{u} = \begin{cases} \mathfrak{D}_{\mathbf{A}^h,\Phi,m}\mathbf{u} \text{ on } \mathbb{R}^n \diagdown \mathbb{L}_{\mathbf{n}^h} \\ \mathfrak{B}_{\mathbb{L}_{\mathbf{n}^h}}\mathbf{u} = a^h_+ \gamma^+_{\mathbb{L}_{\mathbf{n}^h}}\mathbf{u} + a^h_- \gamma^-_{\mathbb{L}_{\mathbf{n}^h}}\mathbf{u} \text{ on } \mathbb{L}_{\mathbf{n}^h} \end{cases} \tag{43}$$

where

$$\mathbb{L}_{\mathbf{n}^h} = \left\{ z \in \mathbb{R}^n : z \cdot \mathbf{n}^h = 0 \right\}, \mathbf{n}^h = \frac{(-1, \nabla f^h)}{\left(1 + |\nabla f^h|^2\right)^{1/2}} \in \mathbb{R}^n,$$

where $\nabla f^h = \lim_{k\to\infty} \nabla f(h_k)$, $a^h_\pm = \Gamma^h \mp i\mathbf{n}^h \cdot \alpha$,
$\mathbf{\Gamma}^h = \lim_{k\to\infty} \Gamma(h_k)$, $h_k \to \eta_\omega \in \Sigma_{\Psi,\infty}$. Note that the hyperplane $\mathbb{L}_{\mathbf{n}^h}$ is the common boundary of the half-spaces

$$\mathbb{R}^n_{\pm,\mathbf{n}^h} = \left\{ x \in \mathbb{R}^n : x \cdot \mathbf{n}^h \gtrless 0 \right\}.$$

The next Theorem give the necessary and sufficient conditions for the interaction operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ to be a Fredholm operator.

## Theorem

*Let $A_j$,$\Phi$, $m \in SO^1(\mathbb{R}^n)$, $\Sigma$ be a slowly oscillating at infinity hypersurface, the Lopatinsky-Shapiro condition hold at every point $x \in \Sigma$. Then*

$$\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} : H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$$

*is a Fredholm operator if and only if all limit operator $\mathbb{D}^h_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ defined by the sequence $h_k \to \eta_\omega \in \mathbb{R}^n_{\Psi,\infty}$ in the topology of $\mathbb{R}^n_\Psi$ are invertible.*

## Corollary

*Let conditions of the above Theorem hold. Then*

$$sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} = \bigcup_h sp\mathcal{D}^h_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$$

*where* $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$, $\mathcal{D}^h_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ *are unbounded operators in* $L^2(\mathbb{R}^n,\mathbb{C}^n)$ *associated with* $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$, $\mathbb{D}^h_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$.

Note that under aditional conditions the spectra of limit operators (43) are

$$sp\mathcal{D}^h_{\mathbf{A},\Phi,a,\mathfrak{B}_\Sigma} = (-\infty, -\left|m^h\right| + \Phi^h)\bigcup[\left|m^h\right| + \Phi^h, +\infty).$$

Under these conditions

$$\begin{aligned} & sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} \\ = \; & (-\infty, \limsup_{x\to\infty}(-\left|m(x)\right| + \Phi(x)]\bigcup[\liminf(\left|m(x)\right| + \Phi(x)), +\infty. \end{aligned}$$